Note 2b Equations of Motion

The Equations of Motion

Here are the most general definitions of the motion variables. These are the differential version of the equations of motion as they use derivatives.

$$\vec{v} = \frac{d\vec{s}}{dt}$$
$$\vec{a} = \frac{d\vec{v}}{dt}$$

In order to find the position function, we integrate both sides of the velocity equations with respect to time to undo the derivative operator. This is just the application of the first fundamental theorem of calculus.

$$\int_{t_i}^{t_f} \vec{v}\,dt = \int_{t_i}^{t_f} \frac{d\vec{s}}{dt}\,dt = \int_{\vec{s}(t_i)}^{\vec{s}(t_f)} d\vec{s} = \vec{s}\left(t_f\right) - \vec{s}\left(t_i\right) = \Delta\vec{s}$$

We can also undo the the time derivative of the velocity.

$$\int_{t_i}^{t_f} \vec{a}\,dt = \int_{t_i}^{t_f} \frac{d\vec{v}}{dt}\,dt = \int_{\vec{v}(t_i)}^{\vec{v}(t_f)} d\vec{v} = \vec{v}\left(t_f\right) - \vec{v}\left(t_i\right) = \Delta\vec{v}$$

Here are the results. These are the integral version of the equations of motion as they use integrals.

$$\Delta\vec{s} = \int_{t_i}^{t_f} \vec{v}\,dt$$
$$\Delta\vec{v} = \int_{t_i}^{t_f} \vec{a}\,dt$$

These are called the equations of motion. They tell us how the position, the velocity, and the acceleration are related and how they evolve over time.

Case of Constant-Acceleration

If the acceleration were constant (including zero), the equations of motion can be simplified. Using the acceleration integral first,

$$\vec{a} = \frac{d\vec{v}}{dt} = constant \quad \Rightarrow \quad \Delta\vec{v} = \int_{t_i}^{t_f} \vec{a}\, dt' = \vec{a}\int_{t_i}^{t_f} dt' = \vec{a}\left(t_f - t_i\right) = \vec{a}\Delta t$$

$$\boxed{\Delta\vec{v} = \vec{a}\Delta t}$$

Using the velocity integral next,

$$\Delta\vec{s} = \int_{t_i}^{t_f} \vec{v}\, dt'$$

But what is the velocity function? We can use acceleration integral for this. We let the final time be a variable instead of a certain value.

$$\vec{a} = \frac{d\vec{v}}{dt} = constant \quad \Rightarrow \quad \vec{v}(t) - \vec{v}_i = \int_{t_i}^{t} \vec{a}\, dt' = \vec{a}\int_{t_i}^{t} dt' = \vec{a}\left(t - t_i\right)$$

$$\vec{v}(t) = \vec{v}_i + \vec{a}\left(t - t_i\right)$$

Using this as the velocity function,

$$\Delta\vec{s} = \int_{t_i}^{t_f}\left[\vec{v}_i + \vec{a}\left(t - t_i\right)\right]dt = \left[\vec{v}_i t + \frac{1}{2}\vec{a}\left(t - t_i\right)^2\right]_{t_1}^{t_f} = \vec{v}_i t_f - \vec{v}_i t_i + \frac{1}{2}\vec{a}\left(t_f - t_i\right)^2 + \frac{1}{2}\vec{a}\left(t_i - t_i\right)^2$$

In short,

$$\boxed{\Delta\vec{s} = \vec{v}_i\Delta t + \frac{1}{2}\vec{a}\Delta t^2}$$

We can also let the initial time be the variable instead. This is an alternate version of the velocity.

$$\vec{v}(t) = -\vec{v}_f + \vec{a}\left(t_f - t\right)$$

The displacement can also be written this way.

$$\boxed{\Delta\vec{s} = \vec{v}_f\Delta t - \frac{1}{2}\vec{a}\Delta t^2}$$

If you don't have or know the elapsed time, you can remove it from the equations above to form an equation of motion with no time in it.

Starting with this equation of motion,

$$\Delta\vec{s} = \vec{v}_i\Delta t + \frac{1}{2}\vec{a}\Delta t^2$$

We take the dot product of the with the acceleration.

$$\vec{a} \cdot (\Delta \vec{s}) = \vec{a} \cdot \left(\vec{v}_i \Delta t + \frac{1}{2} \vec{a} \Delta t^2 \right)$$

$$\vec{a} \cdot \Delta \vec{s} = \vec{a} \cdot \vec{v}_i \Delta t + \vec{a} \cdot \frac{1}{2} \vec{a} \Delta t^2 = (\vec{a} \Delta t) \cdot \vec{v}_i + \frac{1}{2} (\vec{a} \Delta t) \cdot (\vec{a} \Delta t)$$

According to the first equations of motion,

$$\vec{a} \cdot \Delta \vec{s} = (\vec{a} \Delta t) \cdot \vec{v}_i + \frac{1}{2} (\vec{a} \Delta t) \cdot (\vec{a} \Delta t) = \Delta \vec{v} \cdot \vec{v}_i + \frac{1}{2} \Delta \vec{v} \cdot \Delta \vec{v}$$

Move the “2” and multiply out the Δv’s.

$$2\vec{a} \cdot \Delta \vec{s} = 2\Delta \vec{v} \cdot \vec{v}_i + \Delta \vec{v} \cdot \Delta \vec{v} = 2\vec{v}_f \cdot \vec{v}_i - 2\vec{v}_i \cdot \vec{v}_i + \vec{v}_f^{\,2} + \vec{v}_i^{\,2} - \vec{v}_f \cdot \vec{v}_i - \vec{v}_i \cdot \vec{v}_j$$

Collect terms.

$$2\vec{a} \cdot \Delta \vec{s} = \vec{v}_f^{\,2} - \vec{v}_i^{\,2} \quad \Rightarrow \quad \Delta\left(\vec{v}^{\,2}\right) = 2\vec{a} \cdot \Delta \vec{s}$$

This is the fourth equation of motion, the one with no time involved.

$$\boxed{\Delta\left(\vec{v}^{\,2}\right) = 2\vec{a} \cdot \Delta \vec{s}}$$

We can even remove the acceleration from the equations above. This seems strange and works only because the velocity function is linear.

We just take the two displacement equations and add them together.

$$\Delta \vec{s} = \vec{v}_i \Delta t + \frac{1}{2} \vec{a} \Delta t^2 \quad + \quad \Delta \vec{s} = \vec{v}_f \Delta t - \frac{1}{2} \vec{a} \Delta t^2$$

$$2\Delta \vec{s} = \vec{v}_i \Delta t + \vec{v}_f \Delta t$$

$$\boxed{\Delta \vec{s} = \frac{1}{2}\left(\vec{v}_i + \vec{v}_f\right) \Delta t}$$

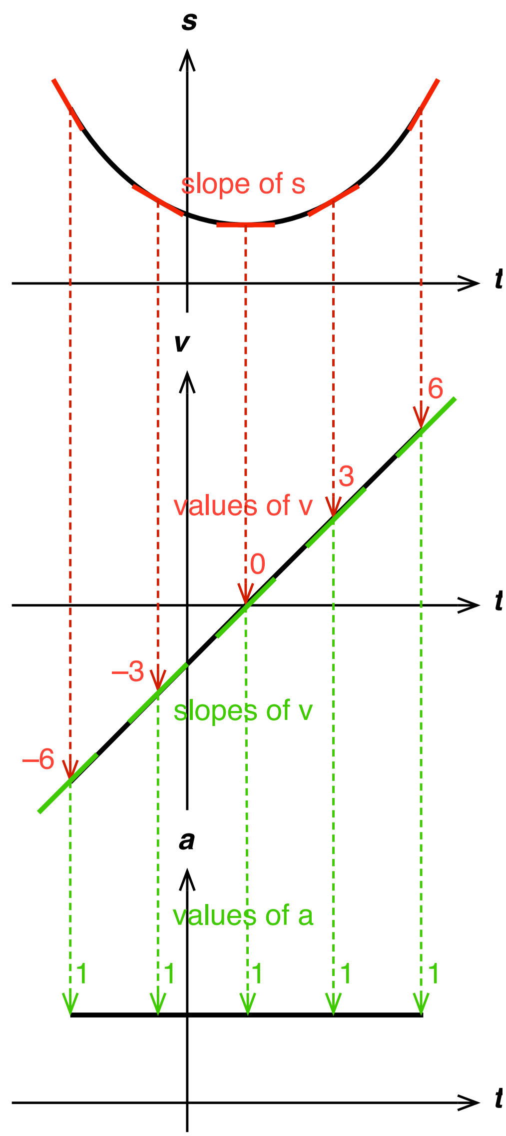
Summary

These are the **constant-acceleration equations of motion**. The note indicates which variable is not involved in the equation.

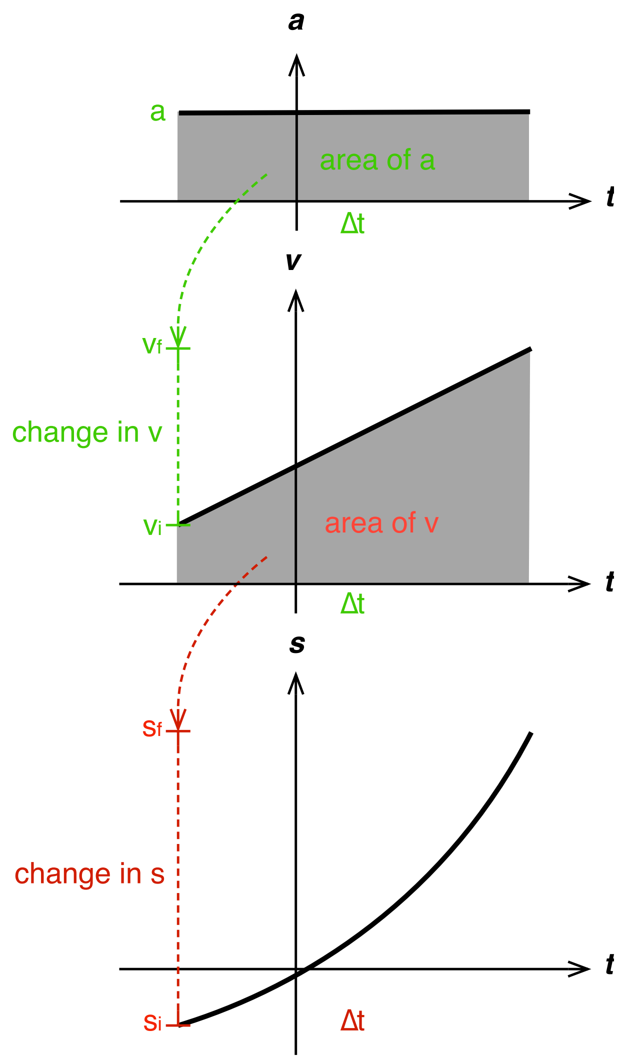
$$\boxed{\begin{array}{ll}
\Delta \vec{v} = \vec{a} \Delta t & no \;\; \Delta \vec{s} \\
\Delta \vec{s} = \vec{v}_i \Delta t + \frac{1}{2} \vec{a} \Delta t^2 & no \;\; \vec{v}_f \\
\Delta \vec{s} = \vec{v}_f \Delta t - \frac{1}{2} \vec{a} \Delta t^2 & no \;\; \vec{v}_i \\
\Delta\left(\vec{v}^{\,2}\right) = 2\vec{a} \cdot \Delta \vec{s} & no \;\; \Delta t \\
\Delta \vec{s} = \frac{1}{2}\left(\vec{v}_f + \vec{v}_i\right) \Delta t & no \;\; \vec{a}
\end{array}}$$

Geometrical Interpretations

From the equations of motion, the velocity is the slope of the position vs time graph and the acceleration is the slope of the velocity vs time graph.



The change in the velocity is the area under the acceleration vs time graph, and the displacement is the area under the velocity vs time graph. For a constant acceleration,



The area under the acceleration graph is a rectangle that is consistent with the constant-acceleration equations of motion.

$$\Delta v = a\Delta t$$

The area under the velocity graph is a rectangle plus a triangle which is also one of our constant-acceleration equations of motion. Here is an equations of motion without the acceleration.

$$\Delta s = v_i\Delta t + \frac{1}{2}\left(v_f - v_i\right)\Delta t = \frac{1}{2}v_i\Delta t + \frac{1}{2}v_f\Delta t = \frac{\left(v_i + v_f\right)}{2}\Delta t$$

$$\Delta s = \frac{\left(v_i + v_f\right)}{2}\Delta t$$

We can also write it in the form from before.

$$\Delta s = v_i\Delta t + \frac{1}{2}\left(v_f - v_i\right)\Delta t = v_i\Delta t + \frac{1}{2}\left(v_i + a\Delta t - v_i\right)\Delta t = v_i\Delta t + \frac{1}{2}a\Delta t^2$$

$$\Delta s = v_i\Delta t + \frac{1}{2}a\Delta t^2$$

The area here is also a rectangle minus a triangle.

$$\Delta s = v_f \Delta t - \frac{1}{2}\left(v_f - v_i\right)\Delta t = v_f \Delta t - \frac{1}{2}\left(v_i + a\Delta t - v_i\right)\Delta t = v_f \Delta t - \frac{1}{2} a \Delta t^2$$

$$\Delta s = v_f \Delta t - \frac{1}{2} a \Delta t^2$$

Velocity Shortcuts

Here are two shortcuts you should always use to keep the problems simpler. First, when the acceleration is zero, the two velocities are the same.

$$\Delta \vec{v} = \vec{a}\Delta t \quad \Rightarrow \quad \Delta \vec{v} = 0 \quad \Rightarrow \quad \vec{v}_f = \vec{v}_i$$

Second, the velocities are the same magnitudes but in opposite directions when the object is at the same position at a second, later time. This means when $\Delta s = 0$.

$$\Delta\left(\vec{v}^{\,2}\right) = 2\vec{a} \cdot \Delta \vec{s} = 0 \quad \Rightarrow \quad \vec{v}_f^{\,2} = \vec{v}_i^{\,2} \quad \Rightarrow \quad \vec{v}_f = \pm \vec{v}_i$$

If the initial velocity is up, then at a later time the object is coming back down, thus, they are opposite of each other.

$$\vec{v}_f = -\vec{v}_i$$

Gravity

When gravity is the only source of acceleration, this situation is called **free-fall**. In this case, the magnitude of the acceleration is called “g”. It is 9.8 m/s^2. The direction of this acceleration at the surface of the Earth points toward the center of the Earth. Locally, we call this downward.

$$\vec{a} = g \cdot \left(-\hat{j}\right)$$