Note 2e Equations of Motion in Two Dimensions

Equations of Motion in Two Dimensions

The independence of perpendicular directions means that the motion in each direction is described by its own set of equations of motion.

For constant acceleration in both directions,

$$\Delta v_x = a_x \Delta t \qquad \Delta v_y = a_y \Delta t$$

$$\Delta x = v_{x,i}\Delta t + \frac{1}{2}a_x \Delta t^2 \qquad \Delta y = v_{y,i}\Delta t + \frac{1}{2}a_y \Delta t^2$$

$$\Delta x = v_{x,f}\Delta t - \frac{1}{2}a_x \Delta t^2 \qquad \Delta y = v_{y,f}\Delta t - \frac{1}{2}a_y \Delta t^2$$

$$\Delta(v_x^2) = 2a_x \Delta x \qquad \Delta(v_y^2) = 2a_y \Delta y$$

$$\Delta x = \frac{1}{2}\left(v_{x,i} + v_{x,f}\right)\Delta t \qquad \Delta y = \frac{1}{2}\left(v_{y,i} + v_{y,f}\right)\Delta t$$

Projectile Motion

In the case of projectile motion, the acceleration in the x direction is zero and the acceleration in the y direction is –g (with down being negative). The acceleration is also constant.

$$\Delta v_x = 0 \qquad \Delta v_y = -g\Delta t$$

$$\Delta x = v_{x,i}\Delta t \qquad \Delta y = v_{y,i}\Delta t - \frac{1}{2}g\Delta t^2$$

$$\Delta x = v_{x,f}\Delta t \qquad \Delta y = v_{y,f}\Delta t + \frac{1}{2}g\Delta t^2$$

$$\Delta(v_x^2) = 0 \qquad \Delta(v_y^2) = -2g\Delta y$$

$$\Delta x = v_{x,i}\Delta t = v_{x,f}\Delta t \qquad \Delta y = \frac{1}{2}\left(v_{y,i} + v_{y,f}\right)\Delta t$$

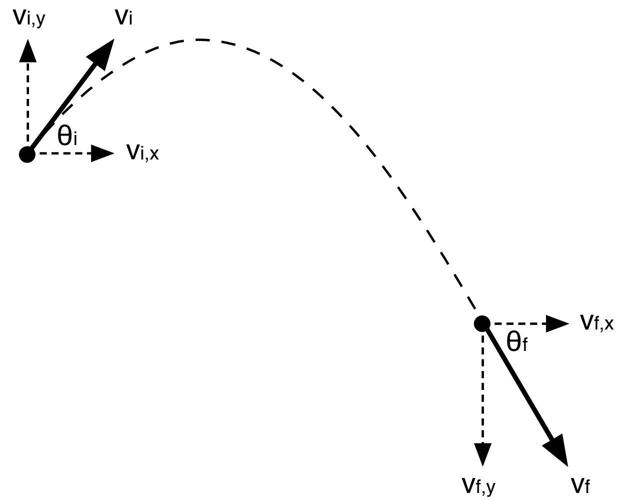
In the x direction, the equations reduce to these two facts.

$$v_{x,i} = v_{x,f}$$

$$\Delta x = v_{x,i}\Delta t = v_{x,f}\Delta t$$

Velocities in Polar Coordinates

Here are the velocities in polar coordinates. To be useful in the equations above, they must be converted to cartesian coordinates.



The components of the velocities are

$$v_{x,i} = v_i \cos\theta_i \qquad v_{y,i} = v_i \sin\theta_i$$

$$v_{x,f} = v_f \cos\theta_f \qquad v_{y,f} = v_f \sin\theta_f$$

Using these for the velocities, free fall in two dimensions looks like this.

$$v_i \cos\theta_i = v_f \cos\theta_f$$

$$\Delta x = v_i \cos\theta_i \Delta t = v_f \cos\theta_f \Delta t$$

$$v_f \sin\theta_f - v_i \sin\theta_i = -g\Delta t$$

$$\Delta y = v_i \sin\theta_i \Delta t - \frac{1}{2} g\Delta t^2$$

$$\Delta y = v_f \sin\theta_f \Delta t + \frac{1}{2} g\Delta t^2$$

$$v_f^2 \sin^2\left(\theta_f\right) - v_i^2 \sin^2\left(\theta_i\right) = -2g\Delta y$$

$$\Delta y = \frac{1}{2}\left(v_i \sin\left(\theta_i\right) + v_f \sin\left(\theta_f\right)\right)\Delta t$$

Projectile Motion Trajectory

The trajectory is the path that an object takes as it travels. In the case of projectile motion, we are looking at the function y(x). We can take the equations of motion and manipulated them to this form. Assume that an object is launched form the origin and that the initial time is zero.

$$y = v_{y,i} t_f - \frac{1}{2} g t_f^2 \quad and \quad x = v_{x,i} t_f$$

Replacing the time with x,

$$t_f = \frac{x}{v_{x,i}}$$

$$y = v_{y,i}\left(\frac{x}{v_{x,i}}\right) - \frac{1}{2} g \left(\frac{x}{v_{x,i}}\right)^2 = -\frac{g}{2v_{x,i}^2} x^2 + \frac{v_{y,i}}{v_{x,i}} x$$

This tells us that the trajectory is a downward-facing parabola. Next, we complete the square.

First, factor out the constant from the x^2 term.

$$y = -\frac{g}{2v_{x,i}^2}\left[x^2 - \frac{2v_{x,i}^2 v_{y,i}}{g v_{x,i}} x\right] = -\frac{g}{2v_{x,i}^2}\left[x^2 - 2\frac{v_{x,i} v_{y,i}}{g} x\right]$$

Next, add and subtract the constant term.

$$y = -\frac{g}{2v_{x,i}^2}\left[x^2 - 2\frac{v_{x,i} v_{y,i}}{g} x + \left(\frac{v_{x,i} v_{y,i}}{g}\right)^2 - \left(\frac{v_{x,i} v_{y,i}}{g}\right)^2\right]$$

$$y = -\frac{g}{2v_{x,i}^2}\left[\left\{x^2 - 2\frac{v_{x,i} v_{y,i}}{g} x + \left(\frac{v_{x,i} v_{y,i}}{g}\right)^2\right\} - \left(\frac{v_{x,i} v_{y,i}}{g}\right)^2\right] = -\frac{g}{2v_{x,i}^2}\left[\left(x - \frac{v_{x,i} v_{y,i}}{g}\right)^2 - \left(\frac{v_{x,i} v_{y,i}}{g}\right)^2\right]$$

Separating the two terms,

$$y = -\frac{g}{2v_{x,i}^2}\left(x - \frac{v_{x,i} v_{y,i}}{g}\right)^2 + \frac{g}{2v_{x,i}^2}\left(\frac{v_{x,i} v_{y,i}}{g}\right)^2 = -\frac{g}{2v_{x,i}^2}\left(x - \frac{v_{x,i} v_{y,i}}{g}\right)^2 + \frac{g}{2v_{x,i}^2}\frac{v_{x,i}^2 v_{y,i}^2}{g^2}$$

$$y = -\frac{g}{2v_{x,i}^2}\left(x - \frac{v_{x,i} v_{y,i}}{g}\right)^2 + \frac{v_{y,i}^2}{2g}$$

This has the form of

$$y = m(x-h)^2 + k$$

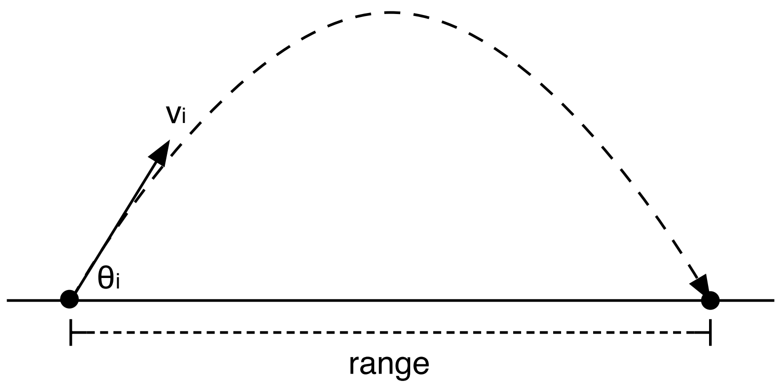
The vertex of the parabola is at h, k. h is half of the range. k is the maximum height. m is the concavity.

$$h = \frac{v_{x,i} v_{y,i}}{g} \quad and \quad k = \frac{v_{y,i}^2}{2g} \quad and \quad m = -\frac{g}{2v_{x,i}^2}$$

Projectile Motion Shortcuts

In the case when an a projectile is launched and lands at the same height, there are several shortcuts that lets us calculate different aspects of the motion.

The **range** of a projectile is the horizontal distance that it travels when it arrives back at the same height as the launch. The **flight time** is the time that a projectile is in the air.



Here are the information for the two directions.

x		y	
$a_x = 0$		$a_y = -10$	
$t_i = 0$	$t_f = ?$	$t_i = 0$	$t_f = ?$
$x_i = 0$	$x_f = R$	$y_i = 0$	$y_f = 0$
$v_{xi} = v_i \cdot \cos(\theta_i)$	$v_{xf} = v_i \cdot \cos(\theta_i)$ $= v_f \cdot \cos(\theta_f)$	$v_{yi} = v_i \cdot \sin(\theta_i)$	$v_{yf} = v_f \cdot \sin(\theta_f)$

Use the y direction to find the flight time.

$$\Delta y = v_{y,i}\Delta t + \frac{1}{2}a_y \Delta t^2 \quad \Rightarrow \quad 0 = v_i \sin\theta_i \Delta t - \frac{1}{2}g\Delta t^2 \quad \Rightarrow \quad \Delta t\left(v_i \sin\theta_i - \frac{1}{2}g\Delta t\right) = 0$$

The solutions are

$$\Delta t = 0 \quad and \quad v_i \sin\theta_i - \frac{1}{2}g\Delta t = 0$$

The flight time is.

$$\Delta t = \frac{2v_i}{g}\sin\theta_i$$

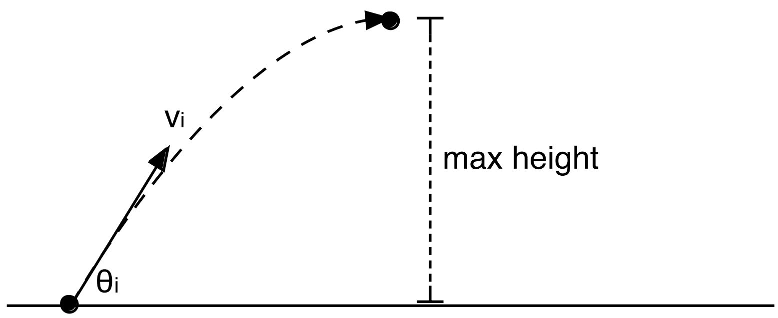
Given this time, the horizontal distance traveled is

$$\Delta x = v_{x,i}\Delta t = v_i \cos\theta_i \frac{2v_i}{g}\sin\theta_i = \frac{v_i^2}{g}2\sin\theta_i\cos\theta_i \quad \Rightarrow \quad \Delta x = \frac{v_i^2}{g}\sin\left(2\theta_i\right)$$

Lastly, the the maximum range is when the following is maximum.

$$\max\left[\sin\left(2\theta_i\right)\right] = 1 \quad \Rightarrow \quad 2\theta_i = 90^\circ \quad \Rightarrow \quad \theta_i = 45^\circ$$

Lastly, we have the maximum height.



For this case, we have that the projectile has a zero vertical velocity at the highest position. Here are the information for the two directions.

x		y	
$a_x = 0$		$a_y = -10$	
$t_i = 0$	$t_f = ?$	$t_i = 0$	$t_f = ?$
$x_i = 0$	$x_f = ?$	$y_i = 0$	$y_f = h$
$v_{xi} = v_i \cdot \cos(\theta_i)$	$v_{xf} = v_i \cdot \cos(\theta_i)$	$v_{yi} = v_i \cdot \sin(\theta_i)$	$v_{yf} = 0$

Note that this has nothing to do with the x direction. I will use the y direction to find the flight time.

$$\Delta v_y = a_y \Delta t \quad \Rightarrow \quad 0 - v_i \sin\theta_i = -g\Delta t \quad \Rightarrow \quad \Delta t = \frac{v_i \sin\theta_i}{g}$$

The height is

$$\Delta y = v_{i,y}\Delta t + \frac{1}{2}a_y \Delta t^2 \quad \Rightarrow \quad h = v_i \sin\theta_i \Delta t - \frac{1}{2}g\Delta t^2 = v_i \sin\theta_i \frac{v_i \sin\theta_i}{g} - \frac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$h = \frac{v_i^2 \sin^2\theta_i}{g} - \frac{1}{2}\frac{v_i^2 \sin^2\theta_i}{g} = \frac{1}{2}\frac{v_i^2 \sin^2\theta_i}{g}$$